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Approaches to learning and teaching

Mathematics

a toolkit for international teachers

Charlie Gilderdale, Alison Kiddle,
Ems Lord, Becky Warren and
Fran Watson

Working with:
Cambridge Assessment
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NRICH

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Series Editors: Paul Ellis and Lauren Harris

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1

Introduction to the series by the editors

1 Approaches to learning and teaching Mathematics

This series of books is the result of close collaboration between Cambridge University Press and Cambridge International Examinations, both departments of the University of Cambridge. The books are intended as a companion guide for teachers, to supplement your learning and provide you with extra resources for the lessons you are planning. Their focus is deliberately not syllabus-specific, although occasional reference has been made to programmes and qualifications. We want to invite you to set aside for a while assessment objectives and grading, and take the opportunity instead to look in more depth at how you teach your subject and how you motivate and engage with your students.

The themes presented in these books are informed by evidence-based research into what works to improve students' learning and pedagogical best practices. To ensure that these books are first and foremost practical resources, we have chosen not to include too many academic references, but we have provided some suggestions for further reading.

We have further enhanced the books by asking the authors to create accompanying lesson ideas. These are described in the text and can be found in a dedicated space online. We hope the books will become a dynamic and valid representation of what is happening now in learning and teaching in the context in which you work.

Our organisations also offer a wide range of professional development opportunities for teachers. These range from syllabus- and topic-specific workshops and large-scale conferences to suites of accredited qualifications for teachers and school leaders. Our aim is to provide you with valuable support, to build communities and networks, and to help you both enrich your own teaching methodology and evaluate its impact on your students.

Each of the books in this series follows a similar structure. In the first chapter, we have asked our authors to consider the essential elements of their subject, the main concepts that might be covered in a school curriculum, and why these are important. The next chapter gives you a brief guide on how to interpret a syllabus or subject guide, and how to plan a programme of study. The authors will encourage you to think too about what is not contained in a syllabus and how you can pass on your own passion for the subject you teach.

The main body of the text takes you through those aspects of learning and teaching which are widely recognised as important. We would like to stress that there is no single recipe for excellent teaching, and that different schools, operating in different countries and cultures, will have strong traditions that should be respected. There is a growing consensus, however, about some important practices and approaches that need to be adopted if students are going to fulfil their potential and be prepared for modern life.

In the common introduction to each of these chapters we look at what the research says and the benefits and challenges of particular approaches. Each author then focuses on how to translate theory into practice in the context of their subject, offering practical lesson ideas and teacher tips. These chapters are not mutually exclusive but can be read independently of each other and in whichever order suits you best. They form a coherent whole but are presented in such a way that you can dip into the book when and where it is most convenient for you to do so.

The final two chapters are common to all the books in this series and are not written by the subject authors. Schools and educational organisations are increasingly interested in the impact that classroom practice has on student outcomes. We have therefore included an exploration of this topic and some practical advice on how to evaluate the success of the learning opportunities you are providing for your students. The book then closes with some guidance on how to reflect on your teaching and some avenues you might explore to develop your own professional learning.

We hope you find these books accessible and useful. We have tried to make them conversational in tone so you feel we are sharing good practice rather than directing it. Above all, we hope that the books will inspire you and enable you to think in more depth about how you teach and how your students learn.

Paul Ellis and Lauren Harris

Series Editors

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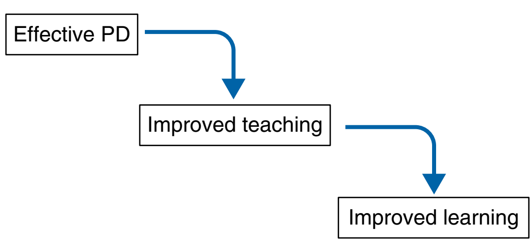
Purpose and context

International research into educational effectiveness tells us that student achievement is influenced most by what teachers do in classrooms. In a world of rankings and league tables we tend to notice performance, not preparation, yet the product of education is more than just examinations and certification. Education is also about the formation of effective learning habits that are crucial for success within and beyond the taught curriculum.

The purpose of this series of books is to inspire you as a teacher to reflect on your practice, try new approaches and better understand how to help your students learn. We aim to help you develop your teaching so that your students are prepared for the next level of their education as well as life in the modern world.

This book will encourage you to examine the processes of learning and teaching, not just the outcomes. We will explore a variety of teaching strategies to enable you to select which is most appropriate for your students and the context in which you teach. When you are making your choice, involve your students: all the ideas presented in this book will work best if you engage your students, listen to what they have to say, and consistently evaluate their needs.



Cognitive psychologists, coaches and sports writers have noted how the aggregation of small changes can lead to success at the highest level. As teachers, we can help our students make marginal gains by guiding them in their learning, encouraging them to think and talk about how they are learning, and giving them the tools to monitor their success. If you take care of the learning, the performance will take care of itself.

When approaching an activity for the first time, or revisiting an area of learning, ask yourself if your students know how to:

- approach a new task and plan which strategies they will use
- monitor their progress and adapt their approach if necessary
- look back and reflect on how well they did and what they might do differently next time.

2 Approaches to learning and teaching Mathematics

Effective learners understand that learning is an active process. We need to challenge and stretch our students and enable them to interrogate, analyse and evaluate what they see and hear. Consider whether your students:

- challenge assumptions and ask questions
- try new ideas and take intellectual risks
- devise strategies to overcome any barriers to their learning that they encounter.

As we discuss in the chapters on **Active learning** and **Metacognition**, it is our role as teachers to encourage these practices with our students so that they become established routines. We can help students review their own progress as well as getting a snapshot ourselves of how far they are progressing by using some of the methods we explore in the chapter on **Assessment for Learning**.

Students often view the subject lessons they are attending as separate from each other, but they can gain a great deal if we encourage them to take a more holistic appreciation of what they are learning. This requires not only understanding how various concepts in a subject fit together, but also how to make connections between different areas of knowledge and how to transfer skills from one discipline to another. As our students successfully integrate disciplinary knowledge, they are better able to solve complex problems, generate new ideas and interpret the world around them.

In order for students to construct an understanding of the world and their significance in it, we need to lead students into thinking habitually about why a topic is important on a personal, local and global scale. Do they realise the implications of what they are learning and what they do with their knowledge and skills, not only for themselves but also for their neighbours and the wider world? To what extent can they recognise and express their own perspective as well as the perspectives of others? We will consider how to foster local and global awareness, as well as personal and social responsibility, in the chapter on **Global thinking**.

As part of the learning process, some students will discover barriers to their learning: we need to recognise these and help students to overcome them. Even students who regularly meet success face their own challenges. We have all experienced barriers to our own learning at some point in our lives and should be able as teachers to empathise and share our own methods for dealing with these. In the chapter on **Inclusive education** we discuss how to make learning accessible for everyone and how to ensure that all students receive the instruction and support they need to succeed as learners.

Some students are learning through the medium of English when it is not their first language, while others may struggle to understand subject jargon even if they might otherwise appear fluent. For all students, whether they are learning through their first language or an additional language, language is a vehicle for learning. It is through language that students access the content of the lesson and communicate their ideas. So, as teachers, it is our responsibility to make sure that language isn't a barrier to learning. In the chapter on **Language awareness** we look at how teachers can pay closer attention to language to ensure that all students can access the content of a lesson.

Alongside a greater understanding of what works in education and why, we as teachers can also seek to improve how we teach and expand the tools we have at our disposal. For this reason, we have included a chapter in this book on **Teaching with digital technologies**, discussing what this means for our classrooms and for us as teachers. Institutes of higher education and employers want to work with students who are effective communicators and who are information literate. Technology brings both advantages and challenges and we invite you to reflect on how to use it appropriately.

This book has been written to help you think harder about the impact of your teaching on your students' learning. It is up to you to set an example for your students and to provide them with opportunities to celebrate success, learn from failure and, ultimately, to succeed.

We hope you will share what you gain from this book with other teachers and that you will be inspired by the ideas that are presented here. We hope that you will encourage your school leaders to foster a positive environment that allows both you and your students to meet with success and to learn from mistakes when success is not immediate. We hope too that this book can help in the creation and continuation of a culture where learning and teaching are valued and through which we can discover together what works best for each and every one of our students.

3

The nature of the subject

Mathematics is a unique subject within the school curriculum. Not only does it underpin many other subject areas, but mathematical truth is fundamentally different from other types of knowledge. Mathematics has a rich and varied history – for as long as people have been reasoning, they have been doing Mathematics. We believe that we have a duty to offer all students the opportunity to engage with rich Mathematics that will allow them to appreciate its unique nature for themselves.

Mathematical truth

Mathematical ideas in some form have existed for thousands of years.

The ancient Greeks were among the first to engage with the sort of thinking that we now recognise as mathematical proof. In ancient Greek society, people spent time debating philosophical ideas. They came to realise that, unlike other types of thought, mathematical truth was resistant to sceptical attack, because once they had proved something, no one else could disprove it. As a consequence, the study of Mathematics became an important part of a classical education. The first proofs were geometrical. Euclid proposed a set of axioms that were widely agreed on, and much of early academic Mathematics consisted of making logical deductions from his starting points in order to prove new theorems.

Humankind has always been fascinated by marking the passage of time. As part of our intrinsic desire to make sense of the world around us, throughout human history we have tried to explain the motion of the stars and planets. We have also taken an interest in scientific questions closer to home, such as why things float, why people get sick, how to build better bridges and how much wine can be held in a barrel. Many developments in the history of Mathematics came about as a result of people needing to explain phenomena, and sharing some of their stories can help to bring Mathematics alive for our students.

Here are some suggestions of historical mathematical figures whose stories might be of interest:

- Archimedes, who solved the problem of working out the density of the king's crown while taking a bath.
- Al-Khwarizmi, who developed methods for solving equations and after whom the word 'algorithm' was named.

- Isaac Newton, who formulated laws that govern motion and gravitational effects, and developed calculus.
- Galileo Galilei, who was imprisoned by the church for expounding the theory of heliocentricity.
- René Descartes, who linked algebra and geometry through his invention of Cartesian coordinates.
- Pierre de Fermat and Blaise Pascal, whose correspondence formed the beginnings of probability theory.

The Mathematics that each of these people did covered a wide range of ideas, but the common thread between them is proof – taking agreed starting points and making deductions to develop new theorems.

If we want our students to work like mathematicians, we need to be explicit about the different types of knowledge seen in a Mathematics classroom. Dave Hewitt introduced the terms ‘arbitrary’, to refer to that which students need to be told, and ‘necessary’, that which they can deduce for themselves.

For example, if you draw a shape with four right angles and four equal sides, the fact that it’s called a square is ‘arbitrary’. Different languages and cultures will use words other than ‘square’ to describe the same object. However, the fact that its diagonals intersect at 90 degrees is a property of a square that students can discover for themselves – it is a ‘necessary’ truth. No matter when or where a square is drawn, its diagonals will always have this property. This type of mathematical truth is universal and it is important that students are offered opportunities to appreciate such universality.

Mathematics as a foundation for other subjects

A teacher challenged her students to come up with a job that didn’t include any Mathematics at all. One student thought he had cracked this with the suggestion ‘priest’, but was met with the response ‘Who would be responsible for the parish accounts then?’ Mathematics is inherent in our daily lives. Whether the context is finance, measurement or making sense of data, students need to be competent and confident in thinking mathematically, in order to live successful lives.

Not only do we want our students to recognise the beauty and truth of Mathematics, we also want to equip them with skills to help them make sense of the world. Students may come into the Mathematics classroom with a passion for Science, Geography, Music, Art or even skateboarding, and this provides us with an opportunity to show them the Mathematics that underpins what they love.

If we want our students to have a well-rounded Mathematics education, we need to give them plenty of experience of doing Mathematics for its own sake, as well as lots of opportunities to apply mathematical thinking to real-world contexts and everyday problem-solving. This balance of pure and applied Mathematics will ensure that our students see Mathematics as useful, worthwhile, interesting and accessible.

As well as sharing stories of pure mathematicians, your students might also be interested to hear about historical figures who used Mathematics in other fields.

- Leonardo da Vinci, artist, inventor, musician, writer…
- Tycho Brahe, who made extremely accurate astronomical observations.
- Isambard Kingdom Brunel, who had visionary ideas about progressing civil engineering in the 19th century.
- Florence Nightingale, who used statistical representations of medical data.

Why not invite your students to research other users of Mathematics in history, to illustrate just how widely the subject has permeated our existence?

Mathematics as a process

We've talked about the nature of mathematical knowledge and the place of Mathematics in relation to other subjects. However, Mathematics can also be considered as a process or set of skills that can be applied in problem-solving contexts.

3 Approaches to learning and teaching Mathematics

Consider the two problems:

'Marbles in a Box' (www.nrich.maths.org/marbles) poses the question: In a three-dimensional version of noughts and crosses, how many winning lines can you make?

'Keep it Simple' (www.nrich.maths.org/keepitsimple) invites students to explore the number of ways in which one-unit fractions can be written as the sum of two different-unit fractions.

$$\frac{1}{6} = \frac{1}{7} + \frac{1}{42}$$
$$\frac{1}{6} = \frac{1}{8} + \frac{1}{24}$$
$$\frac{1}{6} = \frac{1}{9} + \frac{1}{18}$$
$$\frac{1}{6} = \frac{1}{10} + \frac{1}{15}$$

These two problems are taken from very different areas of the curriculum, and at first glance might not seem to have much in common. However, if we consider Mathematics as a way of thinking rather than as a body of knowledge, the two problems are surprisingly similar. Both are concerned with the number of possible solutions to a problem, and both become accessible when students explore the underlying structure.

We often share with students the mathematical content that might be the objective of a particular lesson. This may give the impression that Mathematics IS knowledge. If we also explicitly reference mathematical skills, students can begin to develop their own understanding of what it means to think mathematically.

Here are eight aspects of working mathematically that you may find it helpful to consider:

- exploring and noticing structure
- thinking strategically
- visualising
- representing
- working systematically
- posing questions and making conjectures
- mathematical modelling
- reasoning, convincing and proof.

We will explore these categories in more detail in future chapters.

Summary

The nature of Mathematics can be considered in three broad strands:

- the role of proof in giving mathematical truth a unique status
- the role of Mathematics in supporting other areas of human endeavour
- the processes and skills needed to work mathematically.

A mathematical education that captures all three of these strands has the capacity to convey our excitement about the subject and inspire our students to think like mathematicians.

4 Key considerations

As teachers of Mathematics, there are three key considerations to take into account when planning our lessons. These key considerations need to address students who dislike Mathematics, who lack independence or who are unwilling to use practical resources.

As teachers, we can easily fall into the trap of complaining that students 'can't get started on a problem' or 'they do not like Mathematics'. Our challenge is to turn those comments around, so that students can get started on problems and feel positive about doing and learning Mathematics.

Engaging students

Although Mathematics is a fascinating subject that frequently breaks new ground, pushing the limits of our knowledge and challenging accepted views, many students do not regard it as a dynamic subject. Nevertheless, every lesson offers an opportunity to share our enthusiasm and help students appreciate why so many people find Mathematics such a compelling subject. The following activity offers enormous potential for engaging your students:

LESSON IDEA ONLINE 4.1: CHARLIE'S DELIGHTFUL MACHINE

Many standard questions give just the right amount of information required to solve them. In real life, we often have to sift through information to decide which is the most relevant to solve a given problem. In 'Charlie's Delightful Machine' (www.nrich.maths.org/delightful), students need to go in search of the information and work in a systematic way in order to make sense of the results they gather (see Figure 4.1). They need to try to get all of the lights to switch on, but may discover that it might not always be quite as straightforward as they expect. →

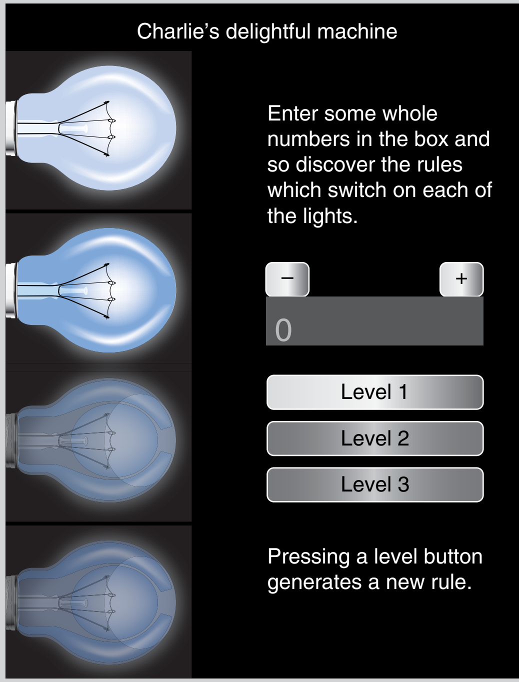


Figure 4.1

Begin the lesson by dividing your board into two columns, one headed with a tick and the other with a cross. Ask students to suggest some numbers, and record each suggestion in the appropriate column according to a rule of your choice. Make it clear to the class that the activity is designed to model scientific enquiry, so they can come up with a hypothesis for your rule, but you will not confirm their hypothesis, only place their numbers in the appropriate column.

Teacher Tip

Here are some suggestions for rules:

- odd numbers
- numbers that are 1 more than multiples of 4
- numbers that are 2 less than multiples of 5
- numbers that are 3 more than multiples of 7.

Once your class have tried the activity with a couple of rules, and are reasonably convinced that their →

hypothesis holds, you can move them on to the main task. Demonstrate the interactive version of the problem, entering a couple of numbers and noting which lights are switched on each time. Make sure that your students understand that more than one light can light up at once, and that each light is governed by its own rule.

Here are some interesting questions that students may ask themselves (or you can encourage them to consider):

1 What is special about rules that light up numbers that:
   - Are all odd?
   - Are all even?
   - Are a mixture of odd and even?
   - Are all multiples of 3? Or 4?
   - Have a last digit of 7?
2 If two lights can be made to switch on together, is there a connection between the rules that light up each individual light and the rule that lights up the pair? Or lights up three at once? Or all four?
3 Sometimes it's impossible to switch a pair of lights on at the same time.

How can you decide when it is impossible?

Multiple representations

Teachers often mistakenly believe that practical resources, such as cubes and number line, are only suitable for lower-attaining students. Nothing could be further from the truth. Students should experience using resources in a variety of ways and have easy access to them in lessons. The effective use of resources allows your students to communicate mathematically, explore alternative approaches to problem-solving and develop as independent students. Many industries make excellent use of modelling to communicate their ideas to a wider audience, such as architecture and aircraft design. All Mathematics classrooms should have a range of easily accessible resources and you should use them frequently in lessons.

LESSON IDEA ONLINE 4.2: FACTORISING WITH MULTILINK

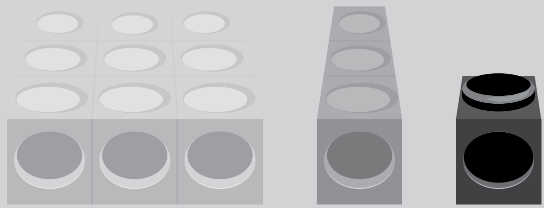


Figure 4.2

When students first meet factorisation, they often don't make the connection between factorising an algebraic expression and breaking a number up into factor pairs. 'Factorising with Multilink' uses a visual representation that allows students to make that connection and discover for themselves the properties necessary for a quadratic expression to factorise. It also relies on a resource that many students initially encountered in their junior school. This activity celebrates the adaptability of resources to support students of all ages and abilities to engage with Mathematics.

To begin this activity, hand out Multilink cubes to your students and ask them to arrange 12 cubes into rectangles. Discuss how the different configurations can be used to illustrate the factors of 12.

In order to introduce the factorisation of algebraic expressions, ask your students to organise themselves into groups of three or four. Select a number base for each group. Using Multilink cubes, give them time to create some squares, sticks and units in their bases. For example, if they are working in base 3 they will need units, sticks of three and squares of nine cubes (see Figure 4.2).

Next, challenge each group to make a rectangle using their different bases. Start by asking them to take 1 square, 3 sticks and 2 units and arrange them to represent an $x^2 + 3x + 2$ rectangle. Students working in base 3 would need:

→

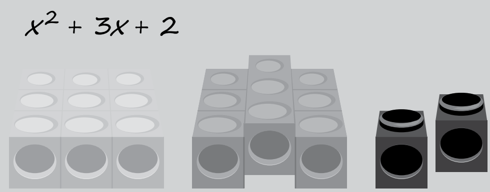


Figure 4.3

Ask the groups to compare their rectangles and ask them what they notice. Have they found an arrangement that works in all bases?

They should agree that the factors of $x^2 + 3x + 2$ are $(x + 1)$ and $(x + 2)$.

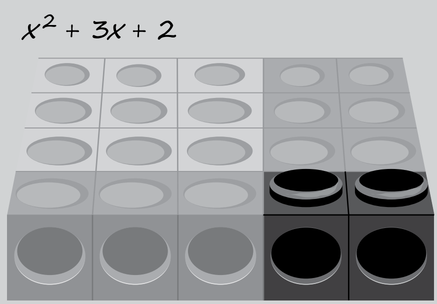


Figure 4.4

You could then ask the groups to make a rectangle using 1 square, 7 sticks and 12 units. Notice that students working in bases 2, 3, 4 and 5 may make arrangements that are not transferable to all other bases. Can they agree on a configuration that works in all bases?

You could also ask the groups to make a rectangle using 1 square, 5 sticks and 8 units. In this case it is possible to make rectangles in bases 2, 3, 4 and 8, but it is not possible to find an arrangement that works in all bases; $x^2 + 5x + 8$ does not factorise.

→

Finally, give students time to explore different combinations of squares, sticks and units and ask them to come up with a quick method for deciding whether a combination of squares, sticks and units can be made into a rectangle that works in all bases (that is, Which quadratic polynomials can be factorised?).

Developing independence

Many students struggle to get started on mathematical problems. In some classrooms, a teacher might set the students a challenge, only to see a wave of hands rise up into the air pleading for help. Such students clearly need to develop some independence in their Mathematics classrooms. Often the cause is learnt helplessness, where the students who have received too much help at an earlier age have become reliant on teachers ever since. Your challenge is to provide activities that develop their independence and confidence while ensuring that you also address the curriculum requirements. In Lesson idea 4.2, the use of Multilink cubes enabled students to access an activity involving different bases. Another way to develop independent learning is by using 'low-threshold, high-ceiling' tasks (see Chapter 10 **Inclusive education**) which aim to be accessible to all students and can also provide increasing levels of challenge.

LESSON IDEA ONLINE 4.3: WHAT'S POSSIBLE?

This problem (www.nrich.maths.org/whatspossible) encourages independent learning while addressing a key area of the curriculum, namely expressing numbers as the difference between two square numbers. An accessible starting point is to ask students for a number between 1 and 30 and then record it on the board as the difference between two squares.

For example, if a student offers 27, you could write $27 = 6^2 - 3^2$. →

Repeat the process for some other numbers such as:

$15 = 8^2 - 7^2$ and $4^2 - 1^2$

and $25 = 5^2 - 0^2$

Challenge individuals or pairs to generate further examples for numbers below 30, and encourage them to record their ideas on the board. Students can collaborate and check that they have all the possible solutions for numbers under 30.

Encourage students to reflect on their results and ask them to consider some of the following questions:

1 Is it possible to write every number as the difference between two square numbers?

2 What is special about the numbers that cannot be expressed as the difference between two square numbers?

3 What is special about the difference between the squares of consecutive numbers?

4 What is special about the difference between the squares of numbers that differ by 2? By 3? By 4? ...

Encourage students to share their ideas and present their findings to the rest of the class. This can lead students to generalise and discover the important identity $a^2 - b^2 = (a + b)(a - b)$.

Students could be challenged to represent the identity diagrammatically.

The initial threshold for 'What's Possible?' is very low, building on some simple initial examples, but, as we have seen, this activity can then lead to interesting generalisations and important discoveries.

Teacher Tip

As a plenary activity, give the class a number and challenge them to find all the ways in which it can be written as the difference between two squares, or convince the the rest of the class that it can't be done.

Summary

Our students may arrive in our Mathematics classroom lacking independent learning skills, displaying a reluctance to use practical resources or failing to appreciate the pleasure that can come from working on mathematical problems. However, we can overcome these obstacles by:

- Modelling the use of resources, we can help students appreciate their power for developing deeper understanding and communicating mathematical ideas.
- Choosing low-threshold, high-ceiling activities we can offer students accessible starting points and suitable follow-up challenges that allow them to work independently at an appropriate level.
- Choosing activities that engage our students, we may find that our students begin to share our enthusiasm for the subject.

Interpreting a syllabus

5

Introduction

A syllabus is a list of requirements that are to be taught on a particular course and usually describes how it is to be examined at the end. It is often set by an exam board or a national body.

Exam boards may use terminology such as subject aims, learning outcomes and assessment objectives. Schools often refer to the syllabus as the subject knowledge and skills that students need in order to satisfy assessment criteria. However, passing an exam is only one part of what we want our students to achieve. We also want them to develop mathematical habits of mind and appreciate the richness of Mathematics. When looking at a syllabus, it is important that we plan schemes of work and units of study that reflect the full range of activities that we consider to be mathematical. This can easily be pushed to one side, particularly when teaching in a 'results-driven' environment. The rest of this chapter will suggest how you can provide a rich mathematical experience for all your students while addressing the requirements of the syllabus.

Weaving the strands of mathematical proficiency

The rope model, proposed in Chapter 4 of '*Adding It Up: Helping Children Learn Mathematics*' (www.nap.edu/read/9822/chapter/6), shows five intertwined strands of mathematical proficiency (Figure 5.1). The authors explain: 'These strands are not independent; they represent different aspects of a complex whole... The five strands are interwoven and interdependent in the development of proficiency in Mathematics. Mathematical proficiency is not a one-dimensional trait, and it cannot be achieved by focusing on just one or two of these strands.' The research behind this model shows that students' proficiency in each individual strand is stronger when we teach in a way that addresses all five strands.

These are the five strands of mathematical proficiency:

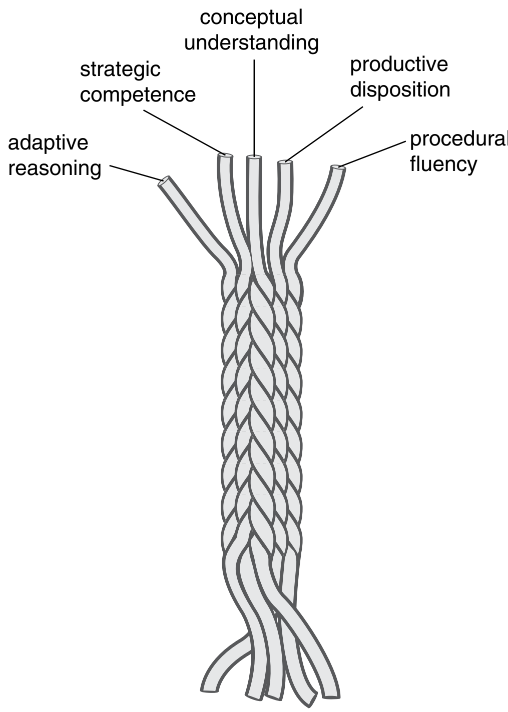


Figure 5.1

Conceptual understanding

Students with conceptual understanding know more than isolated facts and methods. They understand why mathematical ideas are important and can apply them in different contexts. They learn new ideas with understanding, by connecting them to what they already know.

Procedural fluency

Students who are proficient in this strand understand when and how to use procedures, and have the necessary skills to perform them flexibly, accurately and efficiently.

Strategic competence

Strategic competence is required 'to formulate, represent and solve mathematical problems'.

Look at the following strategies to calculate 8 × 15 (Figure 5.2). Are any of these strategies better or worse than any other?

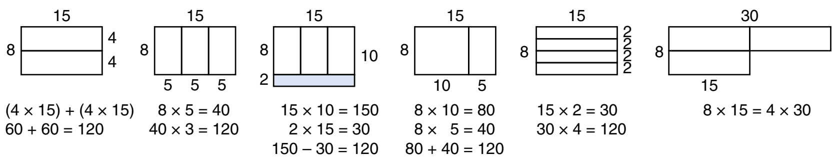


Figure 5.2

Adaptive reasoning

This refers to students' capacity to use logical thought, explanations and justifications. 'In Mathematics, adaptive reasoning is the glue that holds everything together.... In Mathematics, deductive reasoning is used to settle disputes and disagreements.'

Productive disposition

Students with a productive disposition think that Mathematics is useful and worthwhile, believe that with effort and perseverance it can be learnt and applied, and see themselves as capable of making sense of it.

Planning linked activities

Consider the syllabus content statement:

Calculate the mean, median, mode and range for individual and discrete data.

It is tempting to divide this into four lesson objectives, one for each type of average and range, and then address each one separately. Instead, you could plan a series of linked lessons that address all the objectives while developing all aspects of students' mathematical proficiency.

Here are two lesson ideas, based on linked NRICH problems, which address this syllabus content.

LESSON IDEA 5.1: UNEQUAL AVERAGES

(www.nrich.maths.org/11281)

Here’s an interesting set of five numbers:

2, 5, 5, 6, 7

The mean, mode, median and range are all 5.

Try to find other sets of five positive whole numbers where:

Mean = Median = Mode = Range

Stop for a moment before reading on. Try the problem yourself. Continue until you can generate sets with ease.

Now think about the following questions:

1. Which of the five strands have you drawn upon?
2. Did you need to be fluent when working out means, modes and medians?
3. Did you use any strategies to find new sets after you’d found the first few?
4. Did you develop any new insights along the way?
5. Did your reasoning and strategies change as you got to know the problem?
6. How does it feel to know that you can find an infinite set of numbers?

Have a look at this possible follow-up task:

Can you find sets of five positive whole numbers that satisfy the following properties?

Mode	<	Median	<	Mean
Mode	<	Mean	<	Median
Mean	<	Mode	<	Median
Mean	<	Median	<	Mode
Median	<	Mode	<	Mean
Median	<	Mean	<	Mode

Again, stop for a moment before reading on and try the problem yourself.

→

Which of the five strands have you drawn upon this time?

Here is a final challenge:

You may have found that not all of the inequalities can be satisfied with sets of five numbers! Can you explain why?

Can you show that all of them can be satisfied with sets of six numbers?

We like this problem because it combines a need for procedural fluency alongside the development of other skills.

'Wipeout' is a task that could follow on from 'Unequal Averages'.

LESSON IDEA 5.2: WIPEOUT

(www.nrich.maths.org/wipeout)

- Take the numbers 1, 2, 3, 4, 5, 6 and choose one to wipe out.

  For example, you might wipe out 5, leaving you with 1, 2, 3, 4, 6. The mean of what is left is 3.2.

- Is it possible to wipe out one number from 1 to 6, and leave behind an average that is a whole number?
- What about starting with other sets of numbers from 1 to N, where N is even, wiping out just one number and finding the mean?
- Which numbers can be wiped out, so that the mean of what is left is a whole number? Can you explain why?
- What happens when N is odd?

Stop for a moment before reading on and try the problem yourself.

- Would you use this problem with your students?
- Which of the five strands would they require?

More NRICH problems that require students to work with averages can be found at www.nrich.maths.org/averages

Teacher Tip

When planning your scheme of work, create a document with each of the subject content statements from the syllabus. Then populate the document with tasks that address the statement and also develop students' mathematical proficiency.

Selecting a sequence of activities

Here is a longer sequence of linked algebraic activities to consider. As you work through the tasks, think about which of the five strands are being developed as the content is being built up during the sequence.

'Perimeter Expressions'

Start with the NRICH problem 'Perimeter Expressions' (www.nrich.maths.org/perimeterexpressions).

Cut a large rectangular piece of paper in half, take one of the halves and cut that in half again. Continue until you have five rectangles. Label the lengths as shown.

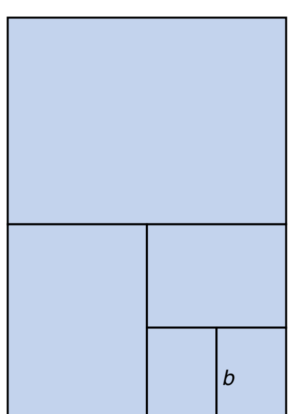


Figure 5.3